Benha University
Final Exam.
Class: *Postgraduate Students*
Subject: *Operations Research*

Faculty of Engineering
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Examiner: *Dr. El-Sayed Badr*

Answer Model

Q1.

a)

1	2	3	4	5	6
×	×	√	√	√	×
Corner point	$\frac{m!}{n!(m-n)!}$				When there is zero element for non- basic variable at objective row (final table)

b)

1	2	3	4
a	c	b	d

Q2:

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Using x_3 as a surplus in the second constraint and x_4 as a slack in the third constraint, the equation form of the problem is given as

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 \qquad\qquad &= 3 \\ 4x_1 + 3x_2 - x_3 \qquad &= 6 \\ x_1 + 2x_2 \qquad + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The third equation has its slack variable, x_4, but the first and second equations do not. Thus, we add the artificial variables R_1 and R_2 in the first two equations and penalize them in the objective function with $MR_1 + MR_2$ (because we are minimizing). The resulting LP is given as

$$\text{Minimize } z = 4x_1 + x_2 + MR_1 + MR_2$$

subject to

$$
\begin{aligned}
3x_1 + x_2 \quad\quad + R_1 \quad &= 3 \\
4x_1 + 3x_2 - x_3 \quad\quad + R_2 &= 6 \\
x_1 + 2x_2 \quad + x_4 \quad\quad &= 4 \\
x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0
\end{aligned}
$$

The associated starting basic solution is now given by $(R_1, R_2, x_4) = (3, 6, 4)$.

From the standpoint of solving the problem on the computer, M must assume a numeric value. Yet, in practically all textbooks, including the first seven editions of this book, M is manipulated algebraically in all the simplex tableaus. The result is an added, and unnecessary, layer of difficulty which can be avoided simply by substituting an appropriate numeric value for M (which is what we do anyway when we use the computer). In this edition, we will break away from the long tradition of manipulating M algebraically and use a numerical substitution instead. The intent, of course, is to simplify the presentation without losing substance.

What value of M should we use? The answer depends on the data of the original LP. Recall that M must be sufficiently large *relative to the original objective coefficients* so it will act as a penalty that forces the artificial variables to zero level in the optimal solution. At the same time, since computers are the main tool for solving LPs, we do not want M to be too large (even though mathematically it should tend to infinity) because potential severe round-off error can result when very large values are manipulated with much smaller values. In the present example, the objective coefficients of x_1 and x_2 are 4 and 1, respectively. It thus appears reasonable to set $M = 100$.

Using $M = 100$, the starting simplex tableau is given as follows (for convenience, the z-column is eliminated because it does not change in all the iterations):

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	-4	-1	0	-100	-100	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

Before proceeding with the simplex method computations, we need to make the z-row consistent with the rest of the tableau. Specifically, in the tableau, $x_1 = x_2 = x_3 = 0$, which yields the starting basic solution $R_1 = 3$, $R_2 = 6$, and $x_4 = 4$. This solution yields $z = 100 \times 3 + 100 \times 6 = 900$ (instead of 0, as the right-hand side of the z-row currently shows). This inconsistency stems from the fact that R_1 and R_2 have nonzero coefficients $(-100, -100)$ in the z-row (compare with the all-slack starting solution in Example 3.3-1, where the z-row coefficients of the slacks are zero).

We can eliminate this inconsistency by substituting out R_1 and R_2 in the z-row using the appropriate constraint equations. In particular, notice the highlighted elements $(= 1)$ in the R_1-row and the R_2-row. Multiplying *each* of R_1-row and R_2-row by 100 and adding the *sum* to the z-row will substitute out R_1 and R_2 in the objective row—that is,

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (100 \times R_1\text{-row} + 100 \times R_2\text{-row})$$

The modified tableau thus becomes (verify!)

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	696	399	−100	0	0	0	900
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
x_4	1	2	0	0	0	1	4

Notice that $z = 900$, which is consistent now with the values of the starting basic feasible solution: $R_1 = 3$, $R_2 = 6$, and $x_4 = 4$.

The last tableau is ready for us to apply the simplex method using the simplex optimality and the feasibility conditions, exactly as we did in Section 3.3.2. Because we are minimizing the objective function, the variable x_1 having the most *positive* coefficient in the z-row (= 696) enters the solution. The minimum ratio of the feasibility condition specifies R_1 as the leaving variable (verify!).

Once the entering and the leaving variables have been determined, the new tableau can be computed by using the familiar Gauss-Jordan operations.

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	0	167	−100	−232	0	0	204
x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1
R_2	0	$\frac{5}{3}$	−1	$-\frac{4}{3}$	1	0	2
x_4	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3

The last tableau shows that x_2 and R_2 are the entering and leaving variables, respectively. Continuing with the simplex computations, two more iterations are needed to reach the optimum: $x_1 = \frac{2}{5}$, $x_2 = \frac{9}{5}$, $z = \frac{17}{5}$ (verify with TORA!).

Note that the artificial variables R_1 and R_2 leave the basic solution in the first and second iterations, a result that is consistent with the concept of penalizing them in the objective function.

Remarks. The use of the penalty M will not force an artificial variable to zero level in the final simplex iteration if the LP does not have a feasible solution (i.e., the constraints are not consistent). In this case, the final simplex iteration will include at least one artificial variable at a positive level. Section 3.5.4 explains this situation.

Two Phase:

We use the same problem in Example 3.4-1.

Phase I

$$\text{Minimize } r = R_1 + R_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 \quad + R_1 \quad &= 3 \\ 4x_1 + 3x_2 - x_3 \quad + R_2 \quad &= 6 \\ x_1 + 2x_2 \quad + x_4 &= 4 \\ x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0 \end{aligned}$$

The associated tableau is given as

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	−1	−1	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
x_4	1	2	0	0	0	1	4

As in the M-method, R_1 and R_2 are substituted out in the r-row by using the following computations:

$$\text{New } r\text{-row} = \text{Old } r\text{-row} + (1 \times R_1\text{-row} + 1 \times R_2\text{-row})$$

The new r-row is used to solve Phase I of the problem, which yields the following optimum tableau (verify with TORA's Iterations ⇒ Two-phase Method):

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	−1	−1	0	0
x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	−1	1	1

Because minimum $r = 0$, Phase I produces the basic feasible solution $x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$, and $x_4 = 1$. At this point, the artificial variables have completed their mission, and we can eliminate their columns altogether from the tableau and move on to Phase II.

Phase II

After deleting the artificial columns, we write the *original* problem as

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$\begin{aligned} x_1 \quad + \tfrac{1}{5}x_3 \quad &= \tfrac{3}{5} \\ x_2 - \tfrac{3}{5}x_3 \quad &= \tfrac{6}{5} \\ x_3 + x_4 &= 1 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Essentially, Phase I is a procedure that transforms the original constraint equations in a manner that provides a starting basic feasible solution for the problem, if one exists. The tableau associated with Phase II problem is thus given as

Basic	x_1	x_2	x_3	x_4	Solution
z	-4	-1	0	0	0
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

Again, because the basic variables x_1 and x_2 have nonzero coefficients in the z-row, they must be substituted out, using the following computations.

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (4 \times x_1\text{-row} + 1 \times x_2\text{-row})$$

The initial tableau of Phase II is thus given as

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$\frac{1}{5}$	0	$\frac{18}{5}$
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

Because we are minimizing, x_3 must enter the solution. Application of the simplex method will produce the optimum in one iteration (verify with TORA).

Q3:

a)

TABLE 5.17 Northwest-Corner Starting Solution

	1	2	3	4	Supply
1	10 5 →	2 10 ↓	20	11	15
2	12	7 5 →	9 15 →	20 5 ↓	25
3	4	14	16	18 10	10
Demand	5	15	15	15	

$$x_{11} = 5, x_{12} = 10$$

$$x_{22} = 5, x_{23} = 15, x_{24} = 5$$

$$x_{34} = 10$$

The associated cost of the schedule is

$$z = 5 \times 10 + 10 \times 2 + 5 \times 7 + 15 \times 9 + 5 \times 20 + 10 \times 18 = \$520$$

TABLE 5.18 Least-Cost Starting Solution

	1	2	3	4	Supply
1	10	(start) 2 15	20	11 0	15
2	12	7	9 15	(end) 20 10	25
3	4 5	14	16	18 5	10
Demand	5	15	15	15	

The resulting starting solution is summarized in Table 5.18. The arrows show the order in which the allocations are made. The starting solution (consisting of 6 basic variables) is $x_{12} = 15, x_{14} = 0, x_{23} = 15, x_{24} = 10, x_{31} = 5, x_{34} = 5$. The associated objective value is

$$z = 15 \times 2 + 0 \times 11 + 15 \times 9 + 10 \times 20 + 5 \times 4 + 5 \times 18 = \$475$$

TABLE 5.19 Row and Column Penalties in VAM

	1	2	3	4		Row penalty
1	10	2	20	11	**15**	$10 - 2 = 8$
2	12	7	9	20	**25**	$9 - 7 = 2$
3	4 **5**	14	16	18	**10**	$14 - 4 = 10$
	5	**15**	**15**	**15**		
Column penalty	$10 - 4 = 6$	$7 - 2 = 5$	$16 - 9 = 7$	$18 - 11 = 7$		

TABLE 5.20 First Assignment in VAM ($x_{31} = 5$)

	1	2	3	4		Row penalty
1	10	2	20	11	**15**	9
2	12	7	9	20	**25**	2
3	4 **5**	14	16	18	**10**	2
	5	**15**	**15**	**15**		
Column penalty	—	5	7	7		

Table 5.20 shows that row 1 has the highest penalty (= 9). Hence, we assign the maximu amount possible to cell (1, 2), which yields $x_{12} = 15$ and simultaneously satisfies both row 1 an column 2. We arbitrarily cross out column 2 and adjust the supply in row 1 to zero.

Continuing in the same manner, row 2 will produce the highest penalty (= 11), and we a: sign $x_{23} = 15$, which crosses out column 3 and leaves 10 units in row 2. Only column 4 is left, an it has a positive supply of 15 units. Applying the least-cost method to that column, we successivel assign $x_{14} = 0$, $x_{34} = 5$, and $x_{24} = 10$ (verify!). The associated objective value for this solution

$$z = 15 \times 2 + 0 \times 11 + 15 \times 9 + 10 \times 20 + 5 \times 4 + 5 \times 18 = \$475$$

This solution happens to have the same objective value as in the least-cost method.

b)

1. A dual variable is defined for each primal (constraint) equation.
2. A dual constraint is defined for each primal variable.
3. The constraint (column) coefficients of a primal variable define the left-hand-side coefficients of the dual constraint and its objective coefficient define the right-hand side.
4. The objective coefficients of the dual equal the right-hand side of the primal constraint equations.

TABLE 4.2 Rules for Constructing the Dual Problem

Primal problem objective*	Dual problem		
	Objective	*Constraints type*	*Variables sign*
Maximization	Minimization	$\geq$	Unrestricted
Minimization	Maximization	$\leq$	Unrestricted

* All primal constraints are equations with nonnegative right-hand side and all the variables are nonnegative.

Q4)

4.4.1 Dual Simplex Algorithm

The crux of the dual simplex method is to start with a better than optimal and infeasible basic solution. The optimality and feasibility conditions are designed to preserve the optimality of the basic solutions while moving the solution iterations toward feasibility.

Dual feasibility condition. The leaving variable, x_r, is the basic variable having the most negative value (ties are broken arbitrarily). If all the basic variables are nonnegative, the algorithm ends.

Dual optimality condition. Given that x_r is the leaving variable, let $\bar{c}_j$ be the reduced cost of nonbasic variable x_j and α_{rj} the constraint coefficient in the x_r-row and x_j-column

of the tableau. The entering variable is the nonbasic variable with $\alpha_{rj} < 0$ that corresponds to

$$\min_{\text{Nonbasic } x_j} \left\{ \left|\frac{\bar{c}_j}{\alpha_{rj}}\right|, \alpha_{rj} < 0 \right\}$$

(Ties are broken arbitrarily.) If $\alpha_{rj} \geq 0$ for all nonbasic x_j, the problem has no feasible solution.

Q5

max $\tilde{z} \underset{\Re}{=} (2, 4, 2, 6)\, x_1 + (2, 6, 1, 3)\, x_2 + (1, 3, 1, 3)\, x_3$

s.t. $x_1 + x_2 + 2x_3 + x_4 = 2$

$2x_1 + 3x_2 + 4x_3 + x_5 = 3$

$6x_1 + 6x_2 + 2x_3 + x_6 = 8$ where : $x_i \geq 0$, i = 1, 2, 3, 4, 5, 6

Step (0): we construct the initial tableau of exterior simplex:

Basis		x_1	x_2	x_3	x_4	x_5	x_6	R.H.S
Z	(-13, -5, 12,4)	(-4,-2, 6, 2)	(-6,-2,3,1)	(-3,-1,3,1)	0	0	0	0
x_4	4	1	1	2	1	0	0	2
x_5	9	2	3	4	0	1	0	3
x_6	14	6	6	2	0	0	1	8

$(\tilde{z}_1 - c_1\ ,\ \tilde{z}_2 - c_2\ ,\ \tilde{z}_3 - c_3) \underset{R}{=}$ ((-2, -4, 6, 2), (-4, -4, 3, 1), (-2, -2, 3, 1)) and $(\gamma_1, \gamma_2, \gamma_3)$ = $(\Re(\gamma_1), \Re(\gamma_2), \Re(\gamma_3))$ = (-8, -9, -5).

Step (1):

J- = {j: $a_{oj} \underset{R}{<} 0$} = {1, 2, 3} $\neq 1$ the Algorithm does not stop.

Step (2): I_+ = {i: $a_{io} > 0$} = {1, 2, 3} $\neq \Phi$ the problem is not unbounded

$$\frac{br}{a_{ro}} = \min\left\{\frac{bi}{aio}, i \in I_+\right\} = \min\left\{\frac{b_1}{a_{10}}, \frac{b_2}{a_{20}}, \frac{b3}{a_{30}}\right\} = \min\left\{\frac{1}{2}, \frac{1}{3}, \frac{4}{7}\right\} = \frac{1}{3} \Rightarrow r = 2$$

Step (3): J_+ = {j: $a_{oj} \underset{R}{>} 0$} = Φ

$$\theta_1 \underset{R}{=} \frac{-a_{ok}}{a_{rk}} \underset{R}{=} \min \left\{ \frac{-aoj}{arj} = j \in J-, arj > 0 \right\} \underset{R}{=} \min \left\{ \frac{-aol}{a_{21}}, \frac{-ao2}{a_{22}}, \frac{-ao3}{a_{23}} \right\} \underset{R}{=} \min \{ (1,2,1,3),$$

$$\left(\frac{2}{3}, 2, \frac{1}{3}, 1 \right), \left(\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4} \right) \}$$

$$R(\theta_1) = \min \left\{ R(2,1,1,3), R\left(\frac{4}{3}, \frac{4}{3}, \frac{1}{3}, 1 \right), R\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \right) \right\} = \min \left\{ 4, 3, \frac{5}{4} \right\} = \frac{5}{4}$$

$\therefore$ k = 3

$$\theta_2 \underset{R}{=} \frac{-aoL}{arL} \underset{R}{=} \min \left\{ \frac{-a_{oj}}{arj} : j \in J_+, arj < o \right\} \Rightarrow R(\theta_2) = \min \{ \Phi \} = \infty$$

$$\Rightarrow R(\theta_1) < R(\theta_2) \Rightarrow \theta_1 \underset{R}{<} \theta_2 \Rightarrow s = k = 3$$

the pivot element is a_{23}

Step (4): the next tableau by pivot element:

Basis		x_1	x_2	x_3	x_4	x_5	x_6	R.H.S
Z	$\left(-5,-3,\frac{42}{4},\frac{30}{4}\right)$	$\left(-3,-1,\frac{13}{2},\frac{7}{2}\right)$	$\left(-2,-3,\frac{15}{4},\frac{13}{4}\right)$	$\tilde{0}$	0	$\left(0,1,\frac{1}{4},\frac{3}{4}\right)$	0	$\left(0,3,\frac{3}{4},\frac{9}{4}\right)$
x_4	$\frac{-1}{2}$	0	$\frac{-1}{2}$	0	1	$\frac{-1}{2}$	0	$\frac{1}{2}$
x_3	$\frac{5}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	0	$\frac{1}{4}$	0	$\frac{3}{4}$
x_6	$\frac{19}{2}$	5	$\frac{9}{2}$	0	0	$\frac{-1}{2}$	1	$\frac{13}{2}$

Step (1): $J_- = \{j: a_{oj} \underset{R}{<} 0\} = \{1, 2\} \Rightarrow$ the Algorithm does not stop.

$$(\gamma_1, \gamma_2, \gamma_5) = (R(\gamma_1), R(\gamma_2), R(\gamma_5)) = \left(\frac{-11}{2}, \frac{-21}{4}, \frac{5}{4} \right)$$

Step (2): $I_+ = \{i: a_{io} > 0\} = \{2,3\} \neq \Phi \Rightarrow$ the problem is bounded

$$\frac{br}{a_{ro}} = \min \left\{ \frac{bi}{a_{io}}, i \in I_+ \right\} = \min \left\{ \frac{b_2}{a_{20}}, \frac{b_3}{a_{30}} \right\} = \min \left\{ \frac{3}{5}, \frac{13}{19} \right\} = \frac{3}{5} \Rightarrow r = 2 <$$

Step (3): $J_+ = \{j: a_{oj} \underset{R}{>} 0\} = \{5\}$

$$\theta_1 \overset{=}{R} \frac{-a_{ok}}{a_{rk}} \underset{R}{=} \min \left\{ \frac{-aoj}{arj} : j \in J, ar_j > 0 \right\} \underset{R}{=} \min \left\{ \frac{-aol}{a_{21}}, \frac{-ao2}{a_{22}} \right\} \underset{R}{=} \min \{ (6,2,7,13) \}, \left(\frac{10}{3}, \frac{10}{3}, \frac{13}{3}, \right)$$

$$R(\theta_1) = \min \left\{ R(6,2,7,13), R\left(\frac{10}{3}, \frac{10}{3}, \frac{13}{3}, 5 \right) \right\} = \min \{11, 7\} = 7 \Rightarrow k = 2$$

$$\theta_2 \underset{R}{=} \frac{-a_{ok}}{a_{rL}} \underset{R}{=} \min \left\{ \frac{-aoj}{arj} : j \in J, arj < 0 \right\} \Rightarrow R(\theta_2) = \min \{ \Phi = \infty \}$$

$\Rightarrow R(\theta_1) < R(\theta_2) \Rightarrow \theta_1 \underset{R}{<} \theta_2 \ \Rightarrow$ s = k = 2

the pivot element is a_{22}

Step (4): the next tableau by pivot element

Basis		x_1	x_2	x_3	x_4	x_5	x_6	R.H.S
Z	$\left(\frac{-4}{3},\frac{2}{3},\frac{26}{3},6\right)$	$\left(\frac{-4}{3},\frac{2}{3},\frac{26}{3},6\right)$	$\tilde{0}$	$\left(\frac{8}{3},4,\frac{13}{3},5\right)$	0	$\left(\frac{2}{3},2,\frac{4}{3},2\right)$	0	(3, 5, 4, 6)
x_4	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$	1	$\frac{-1}{3}$	0	1
x_2	$\frac{2}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	0	$\frac{1}{3}$	0	1
x_6	2	2	0	-6	0	-2	1	2

Step (1): $J_- = \{j : a_{oj} \underset{R}{<} 0\} = \{1\} \neq \Phi \Rightarrow$ the algorithm does not stop .

$$(\gamma_1,\gamma_3,\gamma_5) = (R(\gamma_1), R(\gamma_3), R(\gamma_5)) = (-2,7,3)$$

Step (2): $I_+ = \{i: a_{io} > 0\} = \{1, 2, 3\} \neq \Phi \Rightarrow$ the problem is bounded

$$\text{Ratio test: } \frac{br}{a_{ro}} = \min\left\{\frac{bi}{a_{io}}, i \in I_+\right\} = \min\left\{\frac{b_1}{a_{10}}, \frac{b2}{a_{20}}, \frac{b_3}{a_{30}}\right\} = \min\left\{3, \frac{3}{2}, 1\right\} = 1$$

The index of the entering variable is r = 3

Step (3): $J_+ = \{j: a_{oj} \underset{R}{>} 0\} = \{3,5\}$

$$\theta_1 \underset{R}{=} \frac{-aok}{ark} \underset{R}{=} \min\left\{\frac{-aoj}{arj} : j \in J, arj > 0\right\} \underset{R}{=} \min\left\{\frac{-aol}{a_{31}}\right\} \underset{R}{=} \left(\frac{2}{3}, \frac{-1}{3}, 3, \frac{13}{3}\right)$$

$$R(\theta_1) = R\left(\frac{2}{3}, \frac{-1}{3}, 3, \frac{13}{3}\right) = 1$$

$$\theta_2 \underset{R}{=} \frac{-aol}{arL} \underset{R}{=} \min\left\{\frac{-aoj}{arj} : j \in J, arj < 0\right\} \underset{R}{=} \min\left\{\left(\frac{5}{9}, \frac{5}{9}, \frac{13}{18}, \frac{5}{6}\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 1\right)\right\}$$

$$R(\theta_2) = \min\left\{R\left(\frac{5}{9}, \frac{5}{9}, \frac{13}{18}, \frac{5}{6}\right), R\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 1\right)\right\} = \min\left\{\frac{7}{6}, \frac{3}{2}\right\} = \frac{7}{6}$$

$$R(\theta_1) < R(\theta_2) \Rightarrow s = k = 1$$

the pivot element is a_{31}

Step (4): the next tableau by pivot element

Basis	x_1	x_2	x_3	x_4	x_5	x_6	R.H.S
Z	$\tilde{0}$	0	$(0,\frac{1}{2},1,2)$	0	$(0,\frac{1}{2},1,2)$	0	$(4,6,6,8)$
x_4	0	0	$\frac{5}{3}$	1	0	$\frac{-1}{6}$	$\frac{2}{3}$

x_2	0	$\frac{1}{3}$	$\frac{10}{3}$	0	1	$\frac{-1}{3}$	$\frac{1}{3}$
x_1	1	0	-3	0	-1	$\frac{1}{2}$	1

$(\gamma_3, \gamma_5) = (R(\gamma_3), R(\gamma_5)) = (1, 1)$

Step (1): J₋ : $\{ j : a_{oj} \underset{R}{<} 0 \} = \Phi \Rightarrow$ the Algorithm stops .

the solution is z = R (z) = R $\left(\frac{16}{3}, \frac{10}{3}, 10, \frac{44}{3}\right) = 11$, $x_1 = 1$, $x_2 = \frac{1}{3}$, $x_3 = 0$, $x_4 = \frac{2}{3}$, $x_5 = x_6 = 0$